September $2^{nd}$, 2022.
MMPLS, week 84

# Effective birationality II.

- Effective birationality for Fano varieties with good $\mathbb{Q}$-complements
- Effective birationality for nearly canonical Fano Varieties.
- Boundedness of good $\varepsilon$-lc C.Y pairs (Theorem 1.4).

## Reminder from last week:

**Proposition 4.2** Let $\varepsilon \in \mathbb{R}^{>0}$ and let $\mathcal{P}$ be a bounded set of couples. Then, there is $\delta \in \mathbb{R}^{>0}$ depending only on $\varepsilon$, $\mathcal{P}$ satisfying the following:
Let $(X, B)$ be a projective pair and let $T$ be a reduced divisor on $X$.

Assume:
- $(X, B)$ is $\varepsilon$-lc and $(X, \operatorname{Supp} B + T) \in \mathcal{P}$,
- $L \geq 0$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$,
- $L \sim_{\mathbb{R}} N$ for some $\mathbb{R}$-divisor $N$ supported on $T$, and
- The absolute value of each coefficient of $N$ is at most $\delta$.

Then $(X, B+L)$ is klt.

**Proposition 4.4** Let $d, v \in \mathbb{N}$ and let $\varepsilon \in \mathbb{R}^{>0}$. Then, there exists a number $c \in \mathbb{R}^{>0}$ and a bounded set $\mathcal{P}$ depending only on $d, v, \varepsilon$, satisfying the following.
Assume:
- $X$ is a normal projective variety of dimension $d$,
- $B \geq 0$ is an $\mathbb{R}$-divisor with coefficients in $\{0\} \cup [\varepsilon, \infty)$,
- $M \geq 0$ is a nef $\mathbb{Q}$-divisor such that $|M|$ defines a birational map,
- $M - (K_X + B)$ is pseudo-effective,
- $\operatorname{vol}(M) < v$, and
- If $D$ is a component of $M$, then $\mu_D(B+M) \geq 1$.

Then, there is a projective log smooth $(\overline{X}, \Sigma_{\overline{X}}) \in \mathcal{P}$ and $\varphi : \overline{X} \overset{\text{bir}}{\dashrightarrow} X$ s.t.:
- ) $\operatorname{Supp} \Sigma_{\overline{X}}$ contains the exceptional divisor of $\varphi$ and the bir. transform of $\operatorname{Supp}(B+M)$
- ) $M_{\overline{X}} := \varphi^*(M)$ has coefficients at most $c$.
- ) $\exists\, W \to X$ resolution s.t. $M_W := M|_W \sim A_W + R_W$, with $A_W$ the movable part of $|M_W|$, $|A_W|$ bpf, and for $W \to X' \to X$, $A_{X'} := A_W|_{X'} \sim 0 / \overline{X}$.

**Proposition 4.6** Let $d, v \in \mathbb{N}$ and $\varepsilon', \varepsilon \in \mathbb{R}^{>0}$ with $\varepsilon' < \varepsilon < \frac{1}{2}$. Then, there exists $t \in \mathbb{R}^{>0}$ depending only on $d, v, \varepsilon, \varepsilon'$ satisfying the following.

Assume $X, B, M, \Delta, G, F, \Theta_F, P_F$ are as follows:

- $(X, C)$ is a projective $\varepsilon$-lc pair of dimension $d$,
- $B$ is $\mathbb{R}$-Cartier with coefficients in $\{0\} \cup [\varepsilon, 1-\varepsilon]$,
- $M$ is ample integral divisor and $|M|$ defines a birational map,
- $0 \leq \Delta \sim_{\mathbb{R}} \alpha M$ for some $0 < \alpha < t$,
- $K_X + B + \Delta$ is ample and $M - (K_X + B + \Delta)$ is big,
- $G$ is a member of a covering family of subvarieties of $X$, with normalization $F$,
- $\exists!$ non-klt place of $(X, \Delta)$ whose center is $G$,
- Adjunction Formula gives $K_F + \Theta_F + P_F \sim_{\mathbb{R}} (K_X + \Delta)|_F$, with $P_F \geq 0$, and
- $\mathrm{vol}(M|_F) < v$

Then, for any $0 < L_F \sim_{\mathbb{R}} M|_F$ ; $(F, B|_F + \Theta_F + P_F + t L_F)$ is $\varepsilon'$-lc.

**Proposition 4.8**

Let $d \in \mathbb{N}$ and $\varepsilon, \delta \in \mathbb{R}^{>0}$. Then, there exists a number $p \in \mathbb{N}$ depending only on $d$, $\varepsilon$, and $\delta$ satisfying the following.

Assume:

- $X$ is an $\varepsilon$-lc Fano variety of dimension $d$,
- $m \in \mathbb{N}$ is the smallest number such that $|-mK_{X}|$ defines a bir. map,
- $n \in \mathbb{N}$ is a number such that $\mathrm{vol}(-nK_X) > (2d)^d$, and
- $nK_X + N \sim_{\mathbb{Q}} 0$ for some $\mathbb{Q}$-divisor $N \geq 0$ with coefficients $\geq \delta$.

Then $\frac{m}{n} < p$.

**Idea of Proof** $X_i, m_i, n_i, N_i$, $\frac{m_i}{n_i} \nearrow \infty$.

- create non-klt centers
- $\dim G_i = 0$ ✓✓
- $\dim G_i > 0 \longrightarrow$ lower dimension u.G

$\Leftrightarrow$ bound $\mathrm{vol}(M_i|_{G_i}) \Longrightarrow$ gives lc pairs with large coefficients.

**Proof** **Step 1** $X_i, m_i, n_i, N_i$

Fano $|-m_iK_i|$ birational map.

$\mathrm{vol}(-n_iK_{X_i}) > (2d)^d$

$n_iK_X + N \sim_{\mathbb{Q}} 0$, $\mathrm{coeff}(N_i) \geq \delta$.

$\frac{m_i}{n_i} \nearrow \infty$.

Step 2) $X_i$, covering family $\mathcal{G}_i$ of $X_i$, s.t.

$x_i, y_i \in X_i$, $\Rightarrow \exists\, G_i \in \mathcal{G}_i$ and $0 \le \Delta_i \sim_{\mathbb{Q}} -(n_i+1)K_{X_i}$

s.t. the following holds: $\cdot\,(X_i, \Delta_i)$ is lc near $x_i$, $\exists!$ non-klt place whose center containing $x_i$, the center is $G_i$.

$\cdot\,(X_i, \Delta_i)$ is non-klt near $y_i$.

$d_i := \max(\dim G_i)$.

If $d_i = 0 \Rightarrow 0 \le \Delta_i \sim_{\mathbb{Q}} (-n_i+1)K_{X_i} = \overbrace{\left(\frac{n_i+1}{2n_i+1}\right)}^{<1}\left(-(2n_i+1)K_{X_i}\right)$.

For general $x_i, y_i$, $(X_i, \Delta_i)$ is not klt near $y_i$
$(X_i, \Delta_i)$ is lc near $x_i$ with non-klt center $\{x_i\}$

$(-(2n_i+1)K_{X_i})$ is potentially birational (by definition.)

$\Rightarrow |K_X + (-2n_i+1)K_{X_i}|$ is birational $\quad m_i = 2n_i \Rightarrow \frac{m_i}{n_i} \le 2$.

$l_i :=$ smallest number s.t. $\mathrm{vol}(-l_i K_{X_i}|_{G_i}) > d^d \;\; \forall G_i$, $\;\; \Big|\; \exists G_i$ s.t. $\mathrm{vol}(-(l_i-1)K_{X_i}|_G) \le d^d$

Step 3) $\frac{l_i}{n_i} < a \in \mathbb{N}$.

$\mathrm{vol}((-an_i)K_{X_i}|_{G_i}) > \underline{\underline{d^d}}$.

we can redefine $\Delta'$, $G'$, $\quad 0 \le \Delta' \sim \Delta - (an_i)K_{X_i}$.

$\dim(G') < \dim G$. $\quad = -(\underbrace{(a+1)n_i+1}_{})K_{X_i}$

we can take this family and still $\frac{m_i}{n_i} \nearrow \infty$

with $d_i' < d_i$. $(\Rightarrow\!\Leftarrow)$

$\boxed{\frac{l_i}{n_i} \nearrow \infty}$,

If $\frac{m_i}{l_i} \nearrow \infty$, we can pick $\underline{n_i := l_i}$

$\frac{m_i}{n_i} \nearrow \infty$ but $\frac{n_i}{l_i} = 1$ bounded

$(\Rightarrow \Leftarrow)$

$\boxed{\left(\frac{m_i}{l_i}\right) < \text{constant.}}$

$M_i := -m_i K_{X_i}$.

$\exists G_i$ s.t. $\mathrm{vol}(-(l_i-1)K_{X_i}|_{G_i})$. $(F_i x_i t)$.

$$\mathrm{vol}\left(-m_i K_{X_i}|_G\right) = \underbrace{\left(\frac{m_i}{l_i-1}\right)^d}_{\text{bounded}} \underbrace{\mathrm{vol}\left(-(l_i-1)K_{X_i}|_{G_i}\right)}_{\text{bounded.}}$$

Step 4) $F_i$ normalization of $G_i$, by adjunction.

$$K_{F_i} + \Theta_{F_i} + P_{F_i} \sim_{\mathbb{R}} \underbrace{(K_{X_i} + \Delta_i)}_{\sim -nK_{X_i}}|_{F_i}$$

with $P_{F_i}$ pseff.

general $0 \leq Q \sim -nK_{X_i}$ and add to ge $\Delta_i + Q$.

$\Theta_{F_i}$ would remain the same.

$\Rightarrow P_{F_i}$ big and effective.

Step 6 Apply Prop. 4.6 $\overset{\text{conditions}}{\Longrightarrow}$ $\underbrace{(F_i, \Theta_{F_i} + P_{F_i} + tL_{F_i}) \text{ is } \tfrac{\varepsilon}{2}\text{-lc}}_{\text{for some } t.}$

where $\boxed{L_{F_i} := \frac{m_i}{n_i} \underline{N_{F_i}} \sim_{\mathbb{Q}} -m_i K_{X_i}|_{F_i}}$.

coeff $N > \delta$. Take a component $D \subset \underbrace{\frac{1}{\delta} N_{F_i}}$

$\mu_D(\underbrace{\Theta_{F_i} + \frac{1}{\delta} N_{F_i}}) \geq 1$.

If we have coeff $\frac{1}{\delta}N > 1$. for $D \subset \frac{1}{\delta}N \Rightarrow D_{F_i}$ then $\mu_D(\Theta_{F_i} + \frac{1}{\delta}N_{F_i}) \geq 1$

$\underbrace{\mu_D(\Theta_{F_i}) < 1 - \frac{\varepsilon}{2}}$ $\Rightarrow$ $\mu_D(\frac{1}{\delta} N_{F_i}) \geq \frac{\varepsilon}{2}$

$\mu_D(N_F) \geq \frac{\delta\varepsilon}{2}$, $\Rightarrow$ $\underbrace{\mu_D(t\underline{L_{F_i}})} \geq \frac{\delta\varepsilon}{2} \cdot \boxed{\frac{m_i}{n_i}} \nearrow \infty$

$(\Rightarrow\Leftarrow)$.

□

**Proposition 4.9** Let $d \in \mathbb{N}$ and $\varepsilon, \delta \in \mathbb{R}^{>0}$. Then, there exists a number $m \in \mathbb{N}$ depending only on $d, \varepsilon$, and $\delta$ satisfying the following.
Assume $X$ is an $\varepsilon$-lc Fano variety of dimension $d$ such that $K_X + B \sim_{\mathbb{Q}} 0$ for some $\mathbb{Q}$-divisor $B \geq 0$, with coeff$(B) \geq \delta$.
Then $|-mK_X|$ defines a birational map.

**Idea** $X_i, m_i, B_i$ $\quad m_i \nearrow \infty$.

$\Rightarrow \mathrm{vol}(-m_i K_{X_i}) <$ constant

$\overset{4.4}{\Longrightarrow} \exists$ bounded set ($X_i$ are log birationally bounded)

$\overset{4.2}{\Longrightarrow}$ Some klt pairs (with large coeff).

**Proof** $X_i$, $m_i$, $B_i$.
$X_i$ Fano, $m_i$ smallest s.t. $|-m_i K_{X_i}|$ birational.

$n_i :=$ smallest s.t. $\mathrm{vol}(-nK_{X_i}) > (2d)^d$.

$N_i := n_i B_i \quad \{\mathrm{coeff}\, N_i \geq \delta\}$.

and $n_i K_{X_i} + N \sim_{\mathbb{Q}} 0$

**Prop. 4.8** $\dfrac{m_i}{n_i} < P$.

$$\Rightarrow \mathrm{vol}(\underbrace{-m_i K_{X_i}}_{=M_i}) = \overbrace{\left(\frac{m_i}{n_i - 1}\right)^d}^{\text{bounded.}} \overbrace{\mathrm{vol}(-(n_i - 1)K_{X_i})}^{\leq (2d)^d}$$

Step 2 $\mathrm{vol}(M_i) <$ bounded.

Proposition 4.4 $\Rightarrow (X_i, B_i)$ is log birationally bounded.

$\exists$ bounded set of couples $\mathcal{P}$, constant $c > 0$.

s.t. $\varphi: \overline{X}_i \dashrightarrow X_i$ birational, with log smooth $(\overline{X}_i, \Sigma_{\overline{X}_i})$

- $\boxed{\mathrm{coeff}\, M_{\overline{X}_i} \le C}$

$\mathrm{supp}\, \Sigma_{\overline{X}_i}$ contains exc. divisor of $\varphi$ and bir. transform. of $\mathrm{Supp}(B_i + M_i)$

Step 3 $K_{\overline{X}_i} + \Lambda_{\overline{X}_i} := \varphi^*(K_{X_i})$. $\boxed{X_i \text{ is } \varepsilon\text{-lc}}$

$K_{X_i} + \Delta_i := K_{X_i} + \frac{1}{m_i} M_i \sim_{\mathbb{Q}} 0$.

$\Rightarrow K_{\overline{X}_i} + \Delta_{\overline{X}_i} := K_{\overline{X}_i} + \Lambda_{\overline{X}_i} + \frac{1}{m_i} M_{\overline{X}_i} \sim_{\mathbb{Q}} 0$.

$\mathrm{coeff}(\Lambda_{\overline{X}_i}) \le 1 - \varepsilon$.

$m_i \nearrow \infty$, $\mathrm{coeff}\, \frac{1}{m_i} M_{\overline{X}_i} \searrow 0$.

$\mathrm{coeff}(\Delta_{\overline{X}_i}) \le 1 - \frac{\varepsilon}{2} \Rightarrow \boxed{\Delta_{\overline{X}_i} \le (1 - \frac{\varepsilon}{2}) \Sigma_{\overline{X}_i}}$

$L_i := \frac{1}{\delta} B_i$. $\Rightarrow (X_i, \Delta_i + L_i)$ is not klt

coeff $> 1$

$K_{X_i} + \Delta_i + L_i \sim \underbrace{(1-\frac{1}{\delta}) K_{X_i} + \frac{1}{m_i} M_i}_{\text{isample}}$.

$\Rightarrow (\bar{X}_i, \Delta_{\bar{X}_i} + L_{\bar{X}_i})$ is not sub-klt.

$(\bar{X}_i, (1-\frac{\varepsilon}{2})\Sigma_{\bar{X}_i} + L_{\bar{X}_i})$ is not sub-klt.

$\underline{\text{Prop 4.4}} \Rightarrow (\bar{X}_i, (1-\frac{\varepsilon}{2})\Sigma_{\bar{X}_i} + L_{\bar{X}_i})$ is klt.

$(\Leftarrow \times \Rightarrow)$. $m_i <$ bounded. $\square$

**Proposition 4.11** Let $d \in \mathbb{N}$. Then, there exists numbers $\tau \in (0,1)$ and $m \in \mathbb{N}$ depending only on $d$ satisfying the following. If $X$ is a $\tau$-lc Fano variety of dimension $d$, then $|-mK_X|$ defines a birational map.

**Theorem 1.2** For any $\tau > 0$ there exists $m$.

**Proof** $X_i$, $m_i$, $n_i$, $\varepsilon_i$.

$m_i :=$ min. s.t. $|-m_i K_{X_i}|$ is birational

$M_i := -m_i K_{X_i}$

$n_i :=$ s.t. $\mathrm{vol}(-nK_{X_i}) > (2d)^d$

$m_i \nearrow \infty$, $\varepsilon_i \nearrow 1$, $\dfrac{m_i}{n_i} \nearrow \infty$

We will lift $N$ as in $\boxed{4.8}$ from non-klt centers.

**Step 2** Create a covering family $G_i$

$\dim G_i$, $\boxed{0 \leq \Delta_i \sim -(n_i+1)K_{X_i}.}$

$\dim G_i = 0$ ($\Rightarrow$ $\Leftarrow$)

$l_i :=$ min. s.t. $\mathrm{vol}(-lK_{X_i}|_{G_i}) > d^d$.

Step 3) $l_i, n_i, m_i$. We can Fix $G_i$ s.t.

$$\mathrm{Vol}(-(l_i-1)K_X|_{G_i}) \le d^d.$$

We can (just like in 4.8) (bounding $\frac{l_i}{n_i}, \frac{m_i}{l_i}$)

$$\mathrm{Vol}(-m_i K_{X_i}|G_i) < \underline{\text{bounded}}.$$

Step 4) $F_i$ normalization $G_i$

$$K_{F_i} + \Theta_{F_i} + P_{F_i} \sim_{\mathbb{R}} (K_{X_i} + \Delta_i)|_{F_i}.$$

$P_{F_i}$ big and $\underline{\text{effective}}$.

Step 5) $(F_i, \underbrace{\Theta_{F_i}} + P_F + tL_{F_i})$ is $\varepsilon'$-lc $t>0$.

$\boxed{\varepsilon' < \varepsilon < \varepsilon_i}$ $\quad$ $\mathrm{coeff}(\Theta_F) \subseteq \Psi$ fixed DCC $\underline{\text{set}}$.

We can pick $\varepsilon' \nearrow 1$. $\Rightarrow \mathrm{coeff}(\Theta_F) = 0$.

Step 6) $\underline{\text{Proposition 4.4}}$ can be $\underline{\text{applied}}$.

We get $\mathcal{P}$ bounded set $(F_i, \Sigma_{F_i})$.

$\cdot$ $\mathrm{coeff}(M_{F_i}) \le C$

- $M_{F_i}' \sim A_{F_i}' + R_{F_i}'$, with $A_{F_i}'$ big.
  $|A_{F_i}'|$ bpf.
  $R_{F_i}' \geq 0$ and $A_{F_i}' \sim 0/_{F}$.

Step 7) Reduce to $K_{\bar{F}_i}$ is ps-eff. $\Rightarrow \boxed{K_\sigma(K_{\bar{F}_i}) = 0}$.

- If $K_{\bar{F}_i}$ is not ps-eff., $\exists \lambda$ s.t.

$$\boxed{K_{\bar{F}_i} + \lambda \Sigma_{\bar{F}_i} \text{ is not ps-eff.}}$$

$K_{F_i} + \Lambda_{F_i} := K_{X_i}|_{F_i}$, $\Lambda_{F_i} \leq \Theta_F = 0$, and $(F_i, \Lambda_F)$
sub $\varepsilon_i$-lc

Cas ($X_i$ is $\varepsilon_i$-lc).

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$K_{X_i} + \underline{\frac{1}{m_i}} M_i \sim_{\mathbb{Q}} 0 \Rightarrow K_{\bar{F}_i} + \underbrace{\Lambda_{\bar{F}_i}} + \frac{1}{m} M_{\bar{F}_i} \sim_{\mathbb{Q}} 0$

$\Lambda^{\geq 0}_{F_i}$ is exc./$F_i$ $\Rightarrow$ $\Lambda^{\geq 0}_{\bar{F}_i} \subseteq \Sigma_{\bar{F}_i}$

$$\boxed{\Lambda^{\geq 0}_{\bar{F}_i} \leq \underbrace{(1-\varepsilon_i)}_{\to 0} \Sigma_{\bar{F}_i}}$$

$$\boxed{\frac{1}{m_i} M_{\bar{F}_i} \leq \frac{c}{m_i} \Sigma_{\bar{F}_i}}$$

$\circ$

$$\Lambda^{\geq 0}_{F_i} + \frac{1}{m_i} M_{F_i} \leq \lambda \Sigma_{\bar{F}_i}$$

$$K_{\bar{F}_i} + \Lambda_{\bar{F}_i} + \frac{1}{m_i} M_{\bar{F}_i} \equiv_{\mathbb{Q}} 0.$$

$$\Rightarrow \boxed{K_{\bar{F}_i} + \lambda \Sigma_{\bar{F}_i} \geq 0}$$

it's pseudo-effective.

$\Rightarrow K_{\bar{F}_i}$ is pseudo-effective.

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Step 8) $\kappa_\sigma(K_{\bar{F}_i}) > 0.$

we can get that $\mathrm{vol}(-m_i(1+\ell)K_{X_i}|_F) \nearrow \infty$

Step 9) $\kappa_\sigma(K_{\bar{F}_i}) = 0.$

$\Rightarrow$ we can get $h^0(rK_{F_i}) \neq 0.$

Sections are $\neq 0$. we can get a $\varepsilon$-lc pair with large coefficients.

Sections $= 0$.
$\Rightarrow$ we can lift complement to $\boxed{K_X}$

So $\boxed{4.8} \Rightarrow \frac{m_i}{n_i} <$ Bounded

Step 10 using $\mathrm{vol}(M_i) <$ Bounded.

$\Rightarrow$ Bounded set of couples.

$\Rightarrow$ ($\Rightarrow$ $<$ $\Leftarrow$) $\boxed{\text{with ps-eff thresholds.}}$

$\square$

**Theorem 1.4)** Let $d$ be a natural number, and $\varepsilon$ and $\delta$ be positive real numbers. Consider projective varieties $X$ equipped with a boundary $B$ such that:

- $(X,B)$ is $\varepsilon$-lc of dimension $d$
- $B$ is big and $K_X + B \sim_{\mathbb{R}} 0$, and
- $\operatorname{coeff}(B) \geq \delta$.

Then the set of such $X$ forms a bounded family.

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**Theorem 5.1)** For $X$ as in theorem 1.4

$$\boxed{\operatorname{vol}(-K_X) < v.}$$

**Proof of Theorem 5.1)** Bound volume as before.

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**Proof 1.4)**

As $B$ is Big, we can restrict to $X$ being Fano,

**4.9)** $m$ s.t. $|-mK_X|$ birational

$\mathrm{vol}(-mK_{X_i})$ by 5.1.

4.4 $\Longrightarrow$ says that in such case, we are

$\boxed{\text{log}}$ birationally bounded

By modifying $\varepsilon$ a bit, we get birationally bounded.

$\square$